Transverse modulational instability of partially incoherent soliton stripes

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Based on the Wigner distribution approach, an analysis of the effect of partial incoherence on the transverse instability of soliton structures in nonlinear Kerr media is presented. It is explicitly shown that for a Lorentzian incoherence spectrum the partial incoherence gives rise to a damping which counteracts, and tends to suppress, the transverse instability growth. However, the general picture is more complicated and it is shown that the effect of the partial incoherence depends crucially on the form of the incoherence spectrum. In fact, for spectra with finite rms-width, the partial incoherence may even increase both the growth rate and the range of unstable, transverse wave numbers.

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I. INTRODUCTION

Nonlinear phenomena like self-focusing, collapse, modulational and transverse instabilities of cylindrical light beams are some of the most fundamental consequences of the interplay between linear diffraction and self-phase modulation in nonlinear Kerr media. Various physical mechanisms, which tend to suppress such instabilities, e.g., nonlinear saturation, have been diligently analyzed in a number of works, see references in [1]. These fundamental instability problems have continued to attract attention in connection with new scientific and technical developments. There is currently a strong interest focused on the effects of partial incoherence on different nonlinear instabilities [1–6]. The results of these studies show that the modulational and collapse instabilities in general tend to be suppressed when the waves are partially incoherent. Recently, the effect of partial incoherence on the transverse modulational instability of soliton stripes in nonlinear Kerr media has been investigated, see [2,3]. A soliton stripe is a semi-localized structure, which is of self-trapped soliton form in the x direction, uniform in the y direction, and propagates in the z direction. While a one-dimensional (1D) soliton is resilient to perturbations, the soliton stripe exhibits instability with respect to transverse perturbations, i.e., perturbations in the y direction, see, e.g., Ref. [7]. It has been shown [2,3] that when the stripe is partially incoherent in the y direction, the transverse modulational instability tends to be suppressed and the break-up of the stripe, due to the transverse modulational instability, can be prevented provided the incoherence is sufficiently strong. This behavior is similar to that of the 1D modulational instability. However, analysis of the transverse modulational instability is more complicated than the corresponding analysis in the case of 1D modulational instability. In fact, even in the fully coherent case, the problem of finding the growth rate as a function of the wave number of the perturbations does not have an explicit analytical solution, cf. [7].

In the present work we present an analytical investigation of the effect of partial incoherence on the transverse instability of soliton structures in nonlinear Kerr media. It will be shown that in the case of a Lorentzian incoherence profile, the growth rate of the transverse instability can be expressed simply as the growth rate for the coherent case minus a stabilizing damping rate due to the partial incoherence. However, we also show that the case of a Lorentzian profile represents a very special case and the effect on the growth rate in a general situation depends crucially on the form of the incoherence spectrum, cf. [8]. Using a perturbation approach to the dispersion relation for a general form of the incoherence spectrum, we show analytically that for weak incoherence spectra of finite rms-width, the region of instability always widens and the growth rate is increased in some part of the region. This result agrees well with a recent numerical study of the transverse instability of partially incoherent solitons [3], where the angular spectrum is assumed to have Gaussian form.

II. THE WIGNER APPROACH

Our analysis is based on the Wigner approach, which has been shown to be a convenient tool for analyzing the dynamics of partially incoherent light waves, cf. [1,4–6].

The starting point of the analysis is the nonlinear Schrödinger (NLS) equation for the complex wave field, $\psi(\mathbf{r},z)$, describing the two-dimensional propagation of a partially coherent wave in a diffractive nonlinear Kerr medium,

$$i\frac{\partial\psi}{\partial z} + \frac{1}{2}\nabla_{\perp}^{2}\psi + \langle|\psi|^{2}\rangle\psi = 0, \qquad (1)$$

where the angular brackets $\langle\cdot\rangle$ denote statistical average, z is the distance of propagation, and $\mathbf{r}=(x,y)$ denotes the transverse coordinates. The medium response is here assumed to depend only on the statistically averaged intensity i.e., $I=\langle\psi\psi^{*}\rangle$, and to be of the nonlinear Kerr type. This form of the NLS equation is valid when the medium response time is much larger than the characteristic time of the stochastic intensity fluctuations and yet much shorter than the characteristic time of the wave envelope variation.

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Within the Wigner approach, Eq. (1) is transformed into the Wigner-Moyal equation for the corresponding Wigner function $\rho(\mathbf{r},\mathbf{p},z)$, viz.

$$\frac{\partial\rho}{\partial z} + \mathbf{p}\cdot\frac{\partial\rho}{\partial\mathbf{r}} + 2N(\mathbf{r},z)\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial\mathbf{r}}\cdot\frac{\overrightarrow{\partial}}{\partial\mathbf{p}}\right)\rho(\mathbf{r},\mathbf{p},z) = 0, \quad (2)$$

where the sine operator is defined in terms of its power expansion and the arrows indicate the direction of operation of the derivatives, as explained in more detail in Refs. [4–6]. The Wigner distribution is determined by the stochastic properties of $\psi(\mathbf{r},z)$ and conversely $N=\langle|\psi|^2\rangle=\int\rho d\mathbf{p}$ is the average field intensity. In the present application we consider a background solution in the form of a soliton stripe, i.e., a semi-localized structure, which constitutes a self-trapped soliton form in the x direction, is uniform in the y direction, and propagates in the z direction. This structure is assumed partially incoherent in the y direction. The corresponding intensity and the concomitant Wigner distribution are

$$N_0(x) = \text{sech}^2(x) \quad (3)$$

and

$$\rho_0(x,\mathbf{p}) = \frac{2\sin(2xp_x)}{\sinh(2x)\sinh(\pi p_x)}G(p_y) \equiv R_0(x,p_x)G(p_y), \quad (4)$$

respectively, where $G(p_y)$ characterizes the spectrum of the partial incoherence in the transverse direction. In order to analyze the stability of this background solution, we consider the dynamics of a small perturbation by writing $\rho=\rho_0(x,\mathbf{p})+\rho_1(\mathbf{r},\mathbf{p},z)$, where $\rho_1 \ll \rho_0$. The linear evolution of the small perturbation ρ_1 is then governed by

$$\frac{\partial\rho_1}{\partial z} + \mathbf{p}\cdot\frac{\partial\rho_1}{\partial\mathbf{r}} + 2N_0\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p_x}\right)\rho_1(\mathbf{r},\mathbf{p},z) + 2n_1(\mathbf{r},z)\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial\mathbf{r}}\cdot\frac{\overrightarrow{\partial}}{\partial\mathbf{p}}\right)\rho_0(x,\mathbf{p}) = 0, \quad (5)$$

where $n_1=\int\rho_1 d\mathbf{p}$. When considering the transverse modulational instability, the perturbations can be assumed to be described by harmonic variations, i.e., $n_1(\mathbf{r},z)=n(x)\cos(ky)\exp(\Gamma z)$, where k is the wave number of the transverse perturbation. With this ansatz for the perturbation, Eq. (5) can be rewritten in the compact form

$$\frac{\partial\rho_1}{\partial z} + \mathbf{p}\cdot\frac{\partial\rho_1}{\partial\mathbf{r}} + 2N_0\hat{S}\rho_1 + G_+n\hat{S}R_0 + \frac{G_-}{k}\frac{\partial n_1}{\partial y}\hat{C}R_0 = 0, \quad (6)$$

where we have introduced the operators

$$\hat{S} = \sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p_x}\right), \quad \hat{C} = \cos\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p_x}\right)$$

and used the notations $G_+=G(p_y+k/2)+G(p_y-k/2)$ and $G_-=G(p_y+k/2)-G(p_y-k/2)$. The solution of Eq. (6) can be represented as

$$\rho_1 = [U(x,\mathbf{p})\cos(ky) + V(x,\mathbf{p})\sin(ky)]\exp(\Gamma z), \quad (7)$$

where the unknown functions U and V satisfy

$$\Gamma U + p_x\frac{\partial U}{\partial x} + 2N_0\hat{S}U + kp_yV = -G_+n\hat{S}R_0, \quad (8a)$$

$$\Gamma V + p_x\frac{\partial V}{\partial x} + 2N_0\hat{S}V - kp_yU = G_-n\hat{C}R_0. \quad (8b)$$

This equation system has to be solved subject to the consistency conditions $\int U dp_x dp_y = n(x)$ and $\int V dp_x dp_y = 0$.

III. THE CASE OF LORENTZIAN INCOHERENCE SPECTRUM

For the development of our analysis it is useful to first reconsider the case of a fully coherent wave. The transverse coherence spectrum is then a Dirac delta function i.e., $G(p_y)=\delta(p_y)$. The earlier introduced notations G_+ and G_- now become a sum and a difference, respectively, of two translated delta functions. The p_y dependence of the U and V functions can be expressed in similar manner, i.e., $U=[\delta(p_y+k/2)+\delta(p_y-k/2)]\tilde{u}(\mathbf{r},p_x)$ and $V=[\delta(p_y+k/2)-\delta(p_y-k/2)]\tilde{v}(\mathbf{r},p_x)$. The combinations of delta functions now appearing in Eq. (8) can be shown to be separable, and the resulting system of equations reduces to

$$\hat{L}\tilde{u} - \frac{k^2}{2}\tilde{v} = -n\hat{S}R_0, \quad (9a)$$

$$\hat{L}\tilde{v} + \frac{k^2}{2}\tilde{u} = n\hat{C}R_0, \quad (9b)$$

where we have introduced a p_y-independent operator $\hat{L}$ defined as $\hat{L}=\Gamma+p_x\partial/\partial x+2N_0\hat{S}$. Equation (9) can be combined into a single equation for $\tilde{u}$, which reads

$$\tilde{u} - k^2\hat{P}^{-1}\{\hat{L}\}n\hat{C}R_0 + 2\hat{L}\hat{P}^{-1}\{\hat{L}\}n\hat{S}R_0 = 0, \quad (10)$$

where $\hat{P}^{-1}$ denotes the inverse of the operator $\hat{P}\{\hat{L}\}=k^4/4+(\hat{L})^2$ and curly brackets denote the argument of the operator. The solution of the eigenvalue problem ($\Gamma=\Gamma(0,k)$) cannot be found analytically, and resort must be taken to approximate analytical techniques and/or numerical computations, cf. [9,10]. As an example, a derivation inspired by direct variational methods is given in the Appendix.

With this result in mind for later comparison, we turn back to the partially incoherent problem. In the same way as for the coherent case, we can eliminate the function V in Eq. (8) to obtain

$$\hat{L}^2U + kp_y(kp_yU + G_-n\hat{C}R_0) = -\hat{L}G_+n\hat{S}R_0. \quad (11)$$

Integrating this equation over p_y-space we obtain

$$\int_{-\infty}^{\infty}U dp_y + \int_{-\infty}^{\infty}\hat{A}^{-1}G_-kp_yn\hat{C}R_0 dp_y + \int_{-\infty}^{\infty}\hat{A}^{-1}G_+\hat{L}n\hat{S}R_0 dp_y = 0, \quad (12)$$

where yet another new operator, $\hat{A}$, has been introduced. It is defined as $\hat{A}=[k^2p_y^2+\hat{L}^2]$ and $\hat{A}^{-1}$ denotes its inverse. Since the p_y dependence in the operator $\hat{A}^{-1}$ is multiplicative, some important simplifications can be made. For instance, in the second integral of Eq. (12), the ordering of the terms may be interchanged as $\hat{A}^{-1}G_{-}p_yn\hat{C}R_0=G_{-}p_y\hat{A}^{-1}n\hat{C}R_0$. For the subsequent analysis we need the eigenvalue rather than the operator itself since $\hat{A}^{-1}n\hat{C}R_0=\Sigma_m a_m^{-1}(p_y)c_mn\hat{C}R_0$. The eigenvalue a_m^{-1} corresponding to the operator $\hat{A}^{-1}$ is given by $a_m^{-1}=1/(\lambda_m^2+k^2p_y^2)$ where, in turn, λ_m^2 is the eigenvalue of the $\hat{L}^2$ operator.

We will now assume the incoherence spectrum to have a Lorentzian profile, $G(p_y)=p_0/[\pi(p_y^2+p_0^2)]$, with the characteristic width p_0. This assumption has the important consequence that the integrals appearing in Eq. (12) can be evaluated explicitly to yield

$$\int_{-\infty}^{\infty} a_m^{-1}G_{-}p_ydp_y = \frac{-1}{k[(k/2)^2+(p_0+\lambda_m/k)^2]}, \qquad (13)$$

$$\int_{-\infty}^{\infty} a_m^{-1}G_{+}dp_y = \frac{2(\lambda_m+kp_0)}{k^2\lambda_m[(k/2)^2+(p_0+\lambda_m/k)^2]}. \qquad (14)$$

Thus the dispersion Eq. (12), can be expressed in the following form:

$$\int_{-\infty}^{\infty} Udp_y - k^2\hat{P}^{-1}\{kp_0+\hat{L}\}n\hat{C}R_0 + 2(kp_0+\hat{L})\hat{P}^{-1}\{kp_0 + \hat{L}\}n\hat{S}R_0 = 0. \qquad (15)$$

A comparison of the two dispersion relations [one for the coherent, Eq. (10), and one for the partially incoherent, Eq. (15), case] shows that the only difference between the two is the shift in the argument of the $\hat{P}$ operator; the argument $\hat{L}$ is replaced by $(\hat{L}+kp_0)$ in the partially incoherent case. Equivalently, since $\hat{L}=\Gamma+p_x\partial/\partial x+2N_0\hat{S}$, this implies that $\Gamma(0,k)=\Gamma(p_0,k)+kp_0$, where $\Gamma(p_0,k)$ denotes the growth rate of the partially incoherent case. Thus we finally come to the important conclusion that, for Lorentzian shaped incoherence spectrum, the role of the partial incoherence on the transverse modulational instability of a soliton stripe can be expressed in exactly the same form as for the 1D modulational case [6], viz. simply as a stabilizing damping according to

$$\Gamma(p_0,k) = \Gamma(0,k) - kp_0, \qquad (16)$$

where $\Gamma(0,k)\approx k\sqrt{3-k^2}/2$, cf. the Appendix. This implies two things: the instability is suppressed by the incoherence for all wave numbers in the range $[0,k_c]$, where the cut-off wave number, k_c, is given by $k_c=\sqrt{3-4p_0^2}$ and second, the range of instability decreases monotonously with increasing incoherence. However, this simple monotonously suppressing effect of the partial incoherence on the transverse modulational instability is not of a general nature. An indication of this was found in [3], where numerical investigations were made using Gaussian as well as Lorentzian coherence spectra. Somewhat counter-intuitively it was found that for the case of a Gaussian spectrum, increasing incoherence actually increased the range of modulationally unstable wave numbers and increased the growth rate in part of the unstable region. Only for sufficiently strong incoherence did the unstable wavelength range start to shrink and the growth rate to decrease and to ultimately vanish. Thus, it seems that the properties of the transverse modulational instability depend crucially on the form of the incoherence spectrum. That indeed this is so will be shown analytically in the subsequent paragraph.

IV. RESULTS FOR A GENERAL INCOHERENCE SPECTRUM

In general, a complete analytical solution of Eq. (8) seems impossible to find. However, important information about the properties of the solution can be obtained by considering certain moments of the equations. For this purpose, we integrate the coupled equations for U and V over x and p_x. This yields

$$\Gamma\langle\langle U\rangle\rangle + kp_y\langle\langle V\rangle\rangle = 0, \qquad (17a)$$

$$\Gamma\langle\langle V\rangle\rangle - kp_y\langle\langle U\rangle\rangle = G_{-}\langle\langle nR_0\rangle\rangle, \qquad (17b)$$

where double angular brackets $\langle\langle\cdot\rangle\rangle$ denote integration over x and p_x. The consistency condition for the real part of the perturbation can conveniently be expressed as

$$\int_{-\infty}^{\infty}\langle\langle U\rangle\rangle dp_y = \langle\langle n\delta(p_x)\rangle\rangle. \qquad (18)$$

Thus, solving for $\langle\langle U\rangle\rangle$ from Eq. (17) and inserting this into Eq. (18), we obtain the dispersion relation for the transverse instability of incoherent solutions in the form

$$\int_{-\infty}^{\infty}\frac{kp_yG_{-}}{\Gamma^2+(kp_y)^2}dp_y = -\frac{1}{Q} = -\frac{\langle\langle n\delta(p_x)\rangle\rangle}{\langle\langle nR_0\rangle\rangle}. \qquad (19)$$

We underline that G_{-} is determined by the coherence properties of the soliton background solution, but that the parameter Q may depend on the coherence spectrum. Nevertheless, the result expressed by Eq. (19) is completely general and is valid for arbitrary form of the coherence spectrum. We emphasize that Eq. (19) is of the same form as the dispersion relation for the modulational instability of a partially coherent, but homogeneous, background, cf. [6,8], in which case the parameter Q is easily determined to be $Q=1$. On the other hand, for the transverse instability of a partially incoherent soliton stripe, the proper value of Q cannot be easily found, although we may state that $Q<1$. For the special case of a Lorentzian spectrum studied above, we can take one step further in Eq. (19) to obtain a dispersion relation

$$(\Gamma + kp_0)^2 = (2Q - k^2/2)k^2/2 \qquad (20)$$

where, however, the Q factor still remains to be determined. The analysis of the previous section and the result of the Appendix indicate that $Q\approx 3/4$, independent of the degree of incoherence, i.e., independent of p_0.

In order to pursue this line of analysis for general forms of incoherence spectra, we will assume weak partial incoherence in the sense that the incoherence spectrum is very narrow i.e., $p_0 \ll k$. The integral of Eq. (19) may then be evaluated approximately for any (well-behaved) incoherence spectrum $G(p_y)$. This implies that the function $F(p_y) = kp_y/(\Gamma^2+k^2p_y^2)$ multiplying G_- in the integral can be expanded around the shifted wave numbers $\pm k/2$ to yield

$$\int_{-\infty}^{\infty} F(p_y)[G(p_y+k/2)-G(p_y-k/2)]dp_y \approx -\frac{k^2}{\Phi}+\frac{k^4}{\Phi^2}\left(3-\frac{k^4}{\Phi}\right)p_{\rm rms}^2, \qquad (21)$$

where $\Phi=\Gamma^2+k^4/4$ and we have defined $p_{\rm rms}^2 \equiv \langle x^2\rangle$ as the rms-width of the spectrum

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2 G(x)dx \Bigg/ \int_{-\infty}^{\infty} G(x)dx. \qquad (22)$$

The dispersion relation given by Eq. (19) then becomes

$$\Gamma^2 = k^2\left(Q-\frac{k^2}{4}\right)-\frac{k^4 Q}{\Phi}\left(3-\frac{k^4}{\Phi}\right)p_{\rm rms}^2. \qquad (23)$$

Since the incoherence is assumed weak, we will assume that the dispersion relation given by Eq. (21) may be simplified perturbatively by taking Q equal to its coherent value Q_c and replacing $\Phi=\Gamma^2+k^4/4 \approx k^2 Q_c$ in the incoherently induced correction term. This yields

$$\Gamma^2 \approx k^2\left[\left(Q_c-\frac{k^2}{4}\right)-\left(3-\frac{k^2}{Q_c}\right)p_{\rm rms}^2\right]. \qquad (24)$$

From this approximate expression for the growth rate, we can draw two important conclusions, valid for arbitrary (but narrow) incoherence spectra with finite rms-width: (i) the instability tends to be suppressed for all wave numbers in the range $0<k^2 \leq 3Q_c$, whereas in the region $3Q_c<k^2<4Q_c$, the growth rate is enhanced by the partial incoherence; (ii) the critical (non-zero) wave number, k_c, at which the growth rate goes to zero, increases and is given by $k_c^2 \approx 4(Q_c+p_{\rm rms}^2) \approx 3 + 4p_{\rm rms}^2$. These analytical results agree well with what was obtained by Torres *et al.* [3] using numerical computations.

On the other hand, these results are in contradiction with the results obtained in the previous section for the case of a Lorentzian spectrum. There it was found that (i) the growth rate decreased for all wave numbers, (ii) the cut off wave number, k_c, monotonously decreased with increasing incoherence. The explanation of this apparent contradiction is that the analysis of this section excludes spectra, which, like the Lorentzian, do not have a finite rms-width. A direct implication of this result is that the effect of partial incoherence depends crucially on the form of the incoherence spectrum, even to the extent that in some wavelength range the instability may even be enhanced by the incoherence. As demonstrated in [3], for increasing incoherence, the range of unstable wave numbers first increases, but then eventually shrinks until finally the instability is completely quenched. This complete behavior is outside the range of validity of the perturbation analysis presented in the current section.

V. CONCLUSION

The present analysis has, in some detail, considered the effect of partial incoherence on the transverse modulational instability of soliton stripes in nonlinear Kerr type media. However, it should be possible to generalize our mathematical tool to the case of a saturable nonlinearity. The main problem will then be the soliton structure itself and the instability problem of the coherent case.

We have shown that, for a Lorentzian form of the incoherence spectrum, the effect of partial incoherence on the transverse instability agrees qualitatively with the corresponding result derived for the case of 1D modulational instability; the growth rate decreases monotonously for increasing partial incoherence. However, the Lorentzian form is a very special case in the sense that although it has the nice property of being analytically integrable, it does not have a finite rms-width. Our analysis of general spectra with finite rms-widths shows quite a different qualitative behavior of the growth rate for weak increasing incoherence. The growth rate is found to decrease for transverse wave numbers in the range $0<k<k_*$, but to increase in the complementary range $k_*<k<k_c$, where k_c is the cut off wave number of the instability and k_* is a characteristic transition wave number. In addition, it is found that k_c does in fact increase. These analytical results agree well with numerical simulations performed in [3] as well as with previous analytical work of ours, [8], for the simpler case of the 1D modulational instability.

APPENDIX

The dispersion relation for the transverse modulational instability cannot be determined analytically even in the coherent case and several different approximations have been presented, cf. [9–11]. We will here give a simple, accurate and as far as we know new, approximation using a direct variational approach. Linearization of the two-dimensional coherent NLS equation, given in Eq. (1), around the stationary solution $\psi = \operatorname{sech} x \exp(iz/2)$ gives rise to two coupled equations for the real $u(x)$, and imaginary $v(x)$, parts of the perturbed wave field. Inserting the assumed variations in y and z for the modulational perturbations (i.e., $u,v \propto \exp(iky + \Gamma z)$), these equations become

$$\Gamma u = \hat{L}_1 v, \quad \Gamma v = -\hat{L}_2 u, \qquad (A1)$$

where the operators $\hat{L}_1$ and $\hat{L}_2$ are self-adjoint and defined by

$$\hat{L}_1 = -\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}(1+k^2)-\operatorname{sech}^2 x,$$

$$\hat{L}_2 = \hat{L}_1 - 2\operatorname{sech}^2 x. \qquad (A2)$$

Equation (A1) can be reformulated as a variational problem corresponding to the Lagrangian $\mathcal{L}=\frac{1}{2}v\hat{L}_1 v-\frac{1}{2}u\hat{L}_2 u-\Gamma uv$.

An ansatz is made for the functions u and v as $u=\alpha\phi_2$, $v=\beta\phi_1$, where ϕ_1 and ϕ_2 are trial functions and α and β are the variational parameters. Inserting this ansatz into the variational integral, we find $\langle\mathcal{L}\rangle=\frac{1}{2}\alpha^2\langle\phi_1|\hat{L}_1|\phi_1\rangle-\alpha\beta\Gamma\langle\phi_1|\phi_2\rangle-\frac{1}{2}\beta^2\langle\phi_2|\hat{L}_2|\phi_2\rangle$, where angular brackets $\langle\cdot\rangle$ denote integration over x. The variational equations with respect to α and β give rise to a linear system of equations for these parameters. A nontrivial solution of the system requires its determinant to vanish, giving the following dispersion relation

$$\Gamma^2=-\frac{\langle\phi_2|\hat{L}_2|\phi_2\rangle\langle\phi_1|\hat{L}_1|\phi_1\rangle}{\langle\phi_1|\phi_2\rangle^2}. \tag{A3}$$

With the intuitive choice of the trial functions as equal to the eigenfunctions of the operators $\hat{L}_1$ and $\hat{L}_2$, i.e., $\phi_1=\operatorname{sech}x$ and $\phi_2=\operatorname{sech}^2 x$, respectively, the dispersion relation for the coherent case of the transvere instability becomes

$$\Gamma^2(0,k)\approx\Gamma^2=k^2(3-k^2)\frac{8}{3\pi^2}\approx\frac{k^2}{4}(3-k^2). \tag{A4}$$

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